OBTAINABLE ACCURACY IN THE SOLUTION OF PRACTICAL PROBLEMS BY SMALL-AMPLITUDE WAVE THEORY

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Results obtainable using the theory of progressive waves of small amplitude and certain fundamental solutions relating to waves of finite height are investigated. The theoretical findings are compared with existing experimental data. It is established that the best agreement between the theoretical and experimental profiles of a plane wave is achieved with constructions based on Kozhevnikov's [1] graphs and the equations of motion in the second approximation with respect to the wave height in the form proposed by Mich [2]. The limits within which it is expedient to use the theoretical formulas of the theory of small-amplitude waves and the theory of the second approximation with respect to the wave height are found and proved for the particle velocity, excess pressure, energy flux, and the energy of a single wave.

In solving practical problems connected with wave motion at the surface of a heavy incompressible liquid and its effect on obstacles, it is customary to use the theory of potential waves of infinitely small amplitude. For this form of motion, in the case of progressive waves in water of finite depth, the projections of the velocity on the coordinate axes and the variable part of the local pressure are given by the formulas [3, 4]:

$$v_x = \frac{\sigma h}{2}\frac{\operatorname{ch} k(H+z)}{\operatorname{sh} kH}\cos(\sigma t - kx), \tag{1}$$

$$v_z = \frac{\sigma h}{2}\frac{\operatorname{sh} k(H+z)}{\operatorname{sh} kH}\sin(\sigma t - kx), \tag{2}$$

$$p = \frac{\rho g h}{2}\frac{\operatorname{ch} k(H+z)}{\operatorname{ch} kH}\cos(\sigma t - kx), \tag{3}$$

where h is the height of the wave, λ is the wavelength, τ is the period of the wave, H is the depth of the water, ρ is the density of the water, g is the acceleration of gravity, z the vertical coordinate, taken with a minus sign below the static level, and x the horizontal coordinate at the level of the static horizon.

The linear theory of small-amplitude waves is simple and very convenient for practical purposes. However, in view of the assumptions made, it appears necessary to make a more accurate determination of its limits of applicability and at the same time to show what solutions are to be preferred when the theory fails to give the required accuracy.

It is known that one of the principal indicators of correspondence between wave theory and the phenomenon in question is good agreement between the theoretical and the actual profiles of the agitated surface. In order to make such a comparison, in a wave tank measuring $40 \times 1.0 \times 1.2$ m with glazed side walls we performed a series of experiments to record wave profiles on still and motion-picture film. In addition, we constructed profiles of waves of trochoidal form and small amplitude from Kozhevnikov's data and from the following relations:

1. Stokes [5]:

$$\eta_0 = \frac{h}{2}\cos kx - \frac{kh^2}{16}\frac{\operatorname{ch} kH}{\operatorname{sh}^3 kH}(\operatorname{ch} 2kH + 2)\cos 2kx. \tag{4}$$

2. Nekrasov [6]:

$$x = -\frac{\lambda}{2\pi}\theta - \frac{h}{2}\operatorname{cth}\frac{2\pi H}{\lambda}\sin\theta, \qquad z = \frac{h}{2}\cos\theta. \tag{5}$$

3. Mich [2]:

$$\begin{aligned} x &= x_0 + \frac{h}{2}\frac{\operatorname{ch} k(H+z_0)}{\operatorname{sh} kH}\sin(\sigma t - kx_0) - \frac{kh^2}{16}\frac{\operatorname{sh} 2(\sigma t - kx_0)}{\operatorname{sh}^2 kH}\left[1 - \frac{3}{2}\frac{\operatorname{ch} 2k(H+z_0)}{\operatorname{sh}^2 kH}\right] \\ z &= z_0 + \frac{h}{2}\frac{\operatorname{sh} k(H+z_0)}{\operatorname{sh} kH}\cos(\sigma t - kx_0) + \frac{kh^2}{16}\frac{\operatorname{sh} 2k(H+z_0)}{\operatorname{sh}^2 kH}\left[1 + \frac{3}{2}\frac{\cos 2(\sigma t - kx_0)}{\operatorname{sh}^2 kH}\right] \end{aligned} \tag{6}$$

where η_0 is the elevation of the agitated surface above the static level, and x and z are the wave profile equations.

Kozhevnikov [1] constructed potential wave profiles and determined the characteristics of wave motion by the method of electrohydrodynamic analogies.

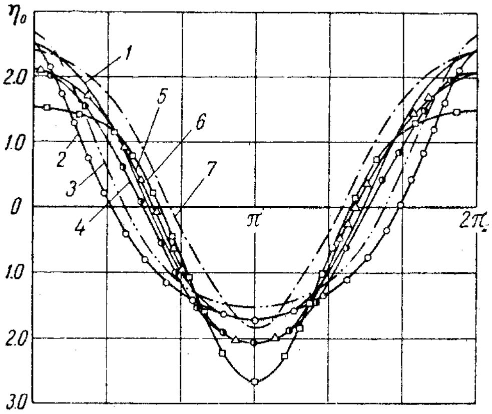

Fig. 1.

An examination of Eqs. (5) and (6) shows that Nekrasov's theory is linear, while Mich takes into account terms up to the second approximation with respect to the height of the wave. Herein lies the principal difference between them. Nekrasov assumed that his theory was valid only for very shallow waves $h/\lambda \leq 1/38$.

By way of example, Fig. 1 shows the experimental and theoretical wave profiles for h = 4.25 cm, λ = 50 cm, H = 9.5 cm, where the curves are numbered: 1) experimental; 2) Kozhevnikov; 3) Mich; 4) linear theory; 5) Nekrasov; 6) Stokes; and 7) trochoidal theory.

In this and other cases it was found that the best approximation to the experimental data is given by constructions based on Kozhevnikov's graphs and computations based on Mich's relations. Unfortunately, Kozhevnikov does not present formulas for the particle velocity and excess pressure corresponding to the wave profiles he obtained. Accordingly, for our purposes we shall use the equations of motion in the form proposed by Mich. These equations have already been used by D. D. Lappo [7], V. V. Khaperskii, and G. G. Metelitsyna [8] for determining the wave pressure on certain types of hydroengineering structures.

The above-mentioned relations have the following form:

$$v_x = \frac{\sigma h}{2}\,\frac{\operatorname{ch} k(H+z)}{\operatorname{sh} kH}\cos(\sigma t - kx) + \frac{3k\sigma h^2}{16}\,\frac{\operatorname{ch} 2k(H+z)}{\operatorname{sh}^4 kH}\cos 2(\sigma t - kx), \tag{7}$$

$$v_z = -\frac{\sigma h}{2}\,\frac{\operatorname{sh} k(H+z)}{\operatorname{sh} kH}\sin(\sigma t - kx) - \frac{3k\sigma h^2}{16}\,\frac{\operatorname{sh} 2k(H+z)}{\operatorname{sh}^4 kH}\sin 2(\sigma t - kx), \tag{8}$$

$$p = \frac{\rho g h}{2}\,\frac{\operatorname{ch} k(H+z)}{\operatorname{ch} kH}\cos(\sigma t - kx) + \frac{3\rho g k h^2 \operatorname{th} kH}{16}\,\frac{\operatorname{ch} 2k(H+z)}{\operatorname{sh}^4 kH}\cos 2(\sigma t - kx) -$$
$$- \frac{\rho g k h^2 \operatorname{th} kH}{16}\,\frac{\operatorname{ch} 2k(H+z)}{\operatorname{sh}^2 kH} - \frac{\rho g k h^2}{16}\,\frac{\operatorname{th} kH}{\operatorname{sh}^2 kH}\cos 2(\sigma t - kx) + \frac{\rho g k h^2}{16}\,\frac{\operatorname{th} kH}{\operatorname{sh}^2 kH}\,. \tag{9}$$

These equations hold true when for $h/\lambda \leq 0.074$ the ratio $H/\lambda \geq 0.132$, while for $H/\lambda \geq 0.146$ there are no restrictions on the steepness of the waves. Formula (9) above was obtained by Biesel [9], using the existing Mich solution, written in a somewhat different form.

Considering the second terms on the right sides of Eqs. (7), (8) and comparing the latter with expressions (1), (2), we see that the numerical values of the particle velocity, averaged over the period of the wave, as obtained from the formulas of the first and second approximations, are the same. On the other hand, for individual moments with respect to the phase of the wave motion the calculated values of the particle velocity may be considerably different, depending on whether formulas (1) and (2) or (7) and (8) are used for the purpose.

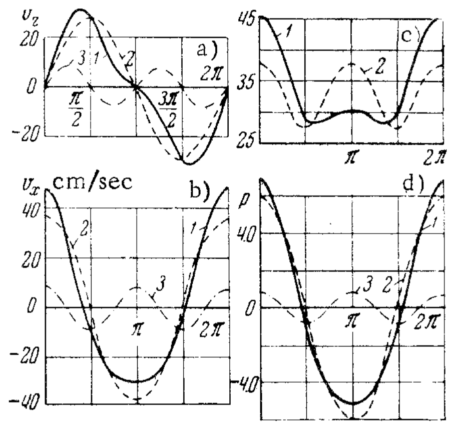

Fig. 2.

This is shown in Fig. 2, which for a complete period gives the results of computations (of the velocities in cm/sec and the pressure p in g/cm^2) based on formulas (1)-(3) and (7)-(9) for z = 0, h = 12.1 cm, λ = 245 cm, H = 36 cm. The individual graphs show: a) vertical projections of the particle velocity; b) horizontal projections of the particle velocity; c) orbital velocities; d) excess wave pressure; 1) second approximation; 2) linear theory; 3) corrections to second approximation. In all the graphs, and especially (c), it is clear that the second-approximation terms have a considerable influence in the individual phases of the period. It is interesting to consider the effect of the relative depth of the water on the terms of second approximation with respect to wave height. We shall do this in relation to the averaged values for the phase of passage of the wave crest, although, as noted above, this averaging leads to a certain drawing together of the results obtained from the theories of waves of small amplitude and finite height. We shall denote averaged values with respect to the depth and the period of the wave by means of brackets, thus: $\langle v_x \rangle$, the auxiliary subscript τ indicating averaging over the period only. From Eqs. (7)-(9), averaging v_x, v_z, and p with respect to the depth, for the first quarter period of the fundamental wave and the first quarter period of the superimposed wave, we get for the wave crest:

$$\langle v_x \rangle = \frac{h\lambda}{\pi\tau H} + \frac{3h^2 \operatorname{ch} kH}{4\tau H \operatorname{sh}^3 kH}\,, \tag{10}$$

$$\langle v_z \rangle = -\frac{h\lambda}{\pi\tau H}\,\frac{(\operatorname{ch} kH - 1)}{\operatorname{sh} kH} - \frac{3h^2 (\operatorname{ch} 2kH - 1)}{8\tau H \operatorname{sh}^4 kH}\,, \tag{11}$$

$$\langle p \rangle = \frac{\rho g h \operatorname{th} kH}{\pi kH} + \frac{3\rho g h^2}{8\pi H \operatorname{sh}^2 kH} - \frac{\rho g h^2}{16H} - \frac{\rho g k h^2}{4\pi \operatorname{sh} 2kH} + \frac{\rho g k h^2}{8 \operatorname{sh} 2kH}\,. \tag{12}$$

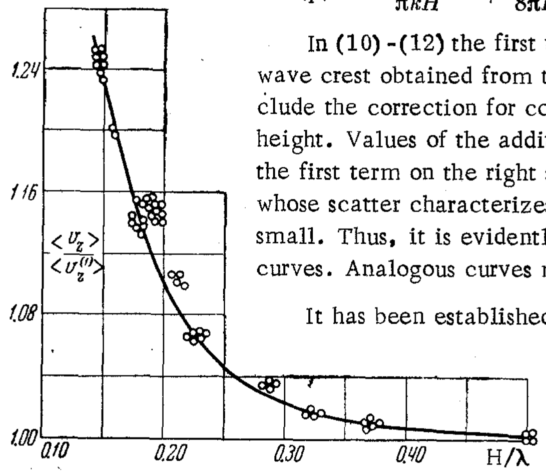

Fig. 3.

In (10)-(12) the first terms on the right sides are equal to the average expressions for the wave crest obtained from the theory of waves of small amplitude, while the following terms include the correction for computation to the second approximation with respect to the wave height. Values of the additional terms for $\langle v_z \rangle$ from (11) are shown in Fig. 3, where $\langle v_z^{(1)} \rangle$ is the first term on the right side of Eq. (11). On the figure we have plotted the theoretical points, whose scatter characterizes the effect of the steepness of the wave. It is clear that this effect is small. Thus, it is evidently undesirable to allow for this effect by constructing supplementary curves. Analogous curves may be constructed for $\langle v_x \rangle$ and $\langle p \rangle$.

It has been established that for a large relative depth the additional terms in (10)-(12) have low values. In practice they can be neglected if $H/\lambda > 0.20$, since in this case the discrepancy between computations based on the formulas of the first and second approximations is less than 10%.

The theoretical and experimental data on the distribution with respect to depth of the particle velocity in a progressive wave are shown in Fig. 4 for the conditions indicated in the table.

In Fig. 4 the depth of the water below the static level, marked $\nabla 0.0$, is plotted along the ordinate axis, and the corresponding orbital velocity along the axis of abscissas, the curves being constructed for the following conditions: 1) experimental, averaged over the whole period; 2) according to the theory of waves of small amplitude or finite amplitude and averaged with respect to the values for the whole phase (which, as already noted, leads to the same results). Curves 1 and 2 lie close together, which confirms the satisfactory agreement over the entire depth of the experimental and theoretical data averaged over the period of the wave.

Graph	H, cm	h, cm	λ, cm	H/λ
(a)	14.5	6.0	107	0.135
(b)	13.6	4.9	75	0.18
(c)	60.0	5.2	140	0.43

The situation is somewhat different if we consider the distribution with respect to depth of the time-averaged orbital velocities for the crest and trough separately. In this case, according to the theory of small-amplitude waves, in view of the influence of the terms of the second approximation, the velocity changes value in moving around the orbit. For example, according to Mich's solution of the second approximation for the wave crest (Fig. 4, curve 3) we have:

$$\langle v_{x\tau} \rangle = \frac{\sigma h}{\pi}\,\frac{\operatorname{ch} k(H+z)}{\operatorname{sh} kH} + \frac{3k\sigma h^2}{8\pi}\,\frac{\operatorname{ch} 2k(H+z)}{\operatorname{sh}^4 kH}\,, \tag{13}$$

$$\langle v_{z\tau} \rangle = -\frac{\sigma h}{\pi}\,\frac{\operatorname{sh} k(H+z)}{\operatorname{sh} kH} - \frac{3k\sigma h^2}{8\pi}\,\frac{\operatorname{sh} 2k(H+z)}{\operatorname{sh}^4 kH}\,, \tag{14}$$

and, correspondingly, for the trough (curve 4):

$$\begin{aligned} \langle v_{x\tau} \rangle &= \frac{\sigma h}{\pi}\,\frac{\operatorname{ch} k(H+z)}{\operatorname{sh} kH} - \frac{3k\sigma h^2}{8\pi}\,\frac{\operatorname{ch} 2k(H+z)}{\operatorname{sh}^4 kH}\,, \\ \langle v_{z\tau} \rangle &= -\frac{\sigma h}{\pi}\,\frac{\operatorname{sh} k(H+z)}{\operatorname{sh} kH} + \frac{3k\sigma h^2}{8\pi}\,\frac{\operatorname{sh} 2k(H+z)}{\operatorname{sh}^4 kH}. \end{aligned} \tag{15}$$

The difference in the values of the orbital velocities at the crest and in the trough of the wave is particularly marked when $H/\lambda \le 0.2$, which is clearly discernible on comparing curves 3 and 4 (Fig. 4).

It is also useful to compare the formulas of the first and second approximations in respect to energy. For example, in solving many problems in the wave motion of a liquid it is necessary to determine the energy transported by a progressive wave. For these purposes it is customary to use a formula obtained from the theory of waves of infinitely small amplitude. This has the form [10, 11]:

$$W_c = \frac{\rho g h^2 c}{16}\left(1 + \frac{4\pi H}{\lambda \operatorname{sh}(4\pi H/\lambda)}\right)$$

where c is the rate of propagation of the wave. Taking into account the second approximation, the energy transport equation for a progressive wave can be obtained as follows. As is known from the basic theory of hydrodynamics,

$$W_c = \int_0^{\tau} dt \int_{-H}^{0} p v_x dz \; . \tag{17}$$

Hence, using Eqs. (7) and (9), after certain transformations we get:

$$W_c = \frac{\rho g h^2 c}{16}\left(1 + \frac{4\pi H}{\lambda \operatorname{sh}(4\pi H/\lambda)}\right) + \frac{\rho \sigma^3 h^3}{48\pi k \operatorname{sh}^4(2\pi H/\lambda)} \times \left[2 \operatorname{ch}^2 \frac{2\pi H}{\lambda}\left(3.0 - \operatorname{sh}^2 \frac{2\pi H}{\lambda}\right) - \operatorname{sh}^2 \frac{2\pi H}{\lambda}\right]. \tag{18}$$

In (18) the second term on the right side is a consequence of taking into account the second approximation with respect to the height of the wave.

Only for $H/\lambda = 0.2$ are the results of computations based on formulas (16) and (18) very close, since in this case the additional term in (18) is negligibly small. When $H/\lambda > 0.2$, the data for (16) exceed the values for (18). For example, for $H/\lambda \geq 0.4$ the discrepancy is 12%. When $H/\lambda < 0.2$ the opposite effect is observed, and the energy flux given by (18) becomes distinctly greater than that given by (16). For $H/\lambda = 0.145$ the increase is already 10%, which is appreciable.

At the same time, it is interesting to compare the wave energy transfer with the total energy of a single wave. The kinetic energy for a progressive wave [10, 11]

$$\tau = \frac{1}{2}\rho \int_0^{\lambda} \varphi \frac{\partial \varphi}{\partial z} dx \; . \tag{19}$$

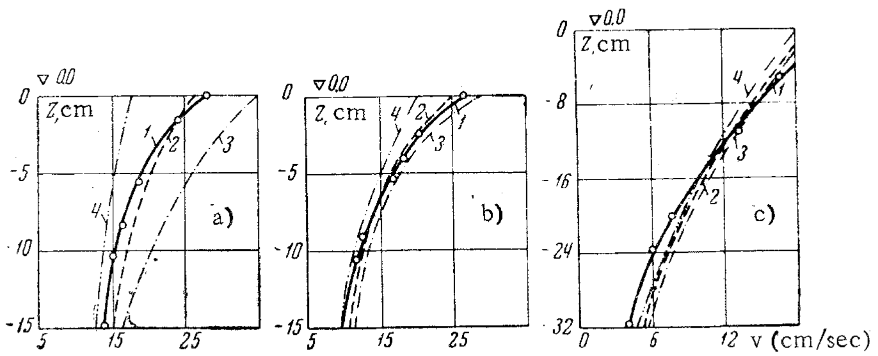

Fig. 4.

For the form of wave motion in question (second approximation with respect to height of wave) the velocity potential

$$\varphi = -\frac{\sigma h}{2k}\frac{\operatorname{ch} k(H+z)}{\operatorname{sh} kH}\sin(\sigma t - kx) - \frac{3\sigma h^2}{32}\frac{\operatorname{ch} 2k(H+z)}{\operatorname{sh}^4 kH}\sin 2(\sigma t - kx). \tag{20}$$

Substituting (20) in (19) for the entire depth of the water we get:

$$\tau = \frac{\rho g h^2 \lambda}{16} + \frac{\rho g \pi^2 h^4}{28.5\lambda}\frac{\operatorname{ch} 2kH}{\operatorname{sh}^6 kH}. \tag{21}$$

The potential energy of a single wave

$$V = \frac{\rho g}{2}\int_0^{\lambda} z^2 dx, \tag{22}$$

where for the case in question

$$z = \frac{h}{2}\cos(\sigma t - kx_0) + \frac{kh^2}{16}\frac{\operatorname{sh} 2kH}{\operatorname{sh}^2 kH}\left[1 + \frac{3\cos 2(\sigma t - kx_0)}{2\operatorname{sh}^2 kH}\right]. \tag{23}$$

In (23) x_0 is the horizontal coordinate of the water particles in the rest state, and not the horizontal coordinate of the wave profile, but for integration within the assumed limits this does not affect the result obtained.

Substituting (23) in (22), we have

$$V = \frac{\rho g h^2 \lambda}{16} + \frac{\rho g \pi^2 h^4}{32\lambda}\operatorname{cth}^2 kH + \frac{\rho g \pi^2 h^4}{14.25\lambda}\frac{\operatorname{cth}^2 kH}{\operatorname{sh}^4 kH}. \tag{24}$$

Whence for the case in question the total energy of a single wave

$$E = \frac{\rho g h^2 \lambda}{8} + \frac{\rho g \pi^2 h^4}{\lambda}\left(\frac{\operatorname{cth}^2 kH}{32} + \frac{\operatorname{sh}^2 kH + 3\operatorname{ch}^2 kH}{28.5\operatorname{sh}^6 kH}\right). \tag{25}$$

The first term on the right side of (25) is equal to the energy of a single wave from the theory of waves of small amplitude. Hence, for waves of finite height of the form in question the energy contained between two vertical lines a distance λ apart is somewhat greater than for waves of infinitely small amplitude.

The results of computations based on (25) give us reason to assume that for the above-mentioned form of waves of finite amplitude the corrections for the increase in energy as compared with waves of small height has a marked effect only when steep waves are propagated at a small relative depth. In these conditions the difference in the energy of a single wave may be as much as 18%. If the wave is shallow, then even at a small relative depth the second term on the right side of (25) is insignificant.

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